

Operator self-similar processes and functional central limit theorems

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Abstract

Let $\{X_k : k \geq 1\}$ be a linear process with values in the separable Hilbert space $L_2(\mu)$ given by $X_k = \sum_{j=0}^{\infty} (j+1)^{-D} \varepsilon_{k-j}$ for each $k \geq 1$, where D is defined by $Df = \{d(s)f(s) : s \in \mathbb{S}\}$ for each $f \in L_2(\mu)$ with $d : \mathbb{S} \rightarrow \mathbb{R}$ and $\{\varepsilon_k : k \in \mathbb{Z}\}$ are independent and identically distributed $L_2(\mu)$ -valued random elements with $E\varepsilon_0 = 0$ and $E\|\varepsilon_0\|^2 < \infty$. We establish sufficient conditions for the functional central limit theorem for $\{X_k : k \geq 1\}$ when the series of operator norms $\sum_{j=0}^{\infty} \|(j+1)^{-D}\|$ diverges and show that the limit process generates an operator self-similar process.

Keywords: linear process; long memory; self-similar process; functional central limit theorem.

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1 Introduction

Self-similar processes are stochastic processes that are invariant in distribution under suitable scaling of time and space. More precisely, let $\xi = \{\xi(t) : t \geq 0\}$ be an \mathbb{R}^q -valued stochastic process defined on some probability space (Ω, \mathcal{F}, P) . The process ξ is said to be self-similar if for any $a > 0$ there exists $b > 0$ such that

$$\{\xi(at) : t \geq 0\} \stackrel{fdd}{=} \{b\xi(t) : t \geq 0\},$$

where $\stackrel{fdd}{=}$ denotes the equality of the finite-dimensional distributions.

Self-similar processes were first studied rigorously by Lamperti [12]. Well-known examples are the Brownian motion and the fractional Brownian motion with Hurst parameter $0 < H < 1$ (in these cases b is equal to $a^{1/2}$ and a^H respectively). We refer to Embrechts and Maejima [6] for the current state of knowledge about self-similar processes and their applications.

Laha and Rohatgi [11] introduced *operator* self-similar processes taking values in \mathbb{R}^q . They extended the notion of self-similarity to allow scaling by a class of matrices. Such processes were later studied by Hudson and Mason [9], Maejima and Mason [15], Lavancier, Philippe, and Surgailis [13], Didier and Pipiras [5] among others.

Matache and Matache [16] consider and study operator self-similar processes valued in (possibly infinite-dimensional) Banach spaces. Let \mathbb{E} denote a Banach space and let $L(\mathbb{E})$ be the algebra of all bounded linear operators on \mathbb{E} . Matache and Matache [16] give the following definition.

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Definition. An operator self-similar process is a stochastic process $\xi = \{\xi(t) : t \geq 0\}$ on \mathbb{E} such that there is a family $\{T(a) : a > 0\}$ in $L(\mathbb{E})$ with the property that for each $a > 0$,

$$\{\xi(at) : t \geq 0\} \stackrel{fdd}{=} \{T(a)\xi(t) : t \geq 0\}.$$

The family $\{T(a) : a > 0\}$ is called the scaling family of operators. If operators $\{T(a) : a > 0\}$ have the particular form $T(a) = a^G I$, where G is some fixed scalar and I is an identity operator, then a stochastic process is called self-similar instead of operator self-similar.

In this paper, we obtain an example of an operator self-similar process with values in the real separable Hilbert space $L_2(\mu) = L_2(\mathbb{S}, \mathcal{S}, \mu)$ of equivalence classes of μ -almost everywhere equal square-integrable functions, where $(\mathbb{S}, \mathcal{S}, \mu)$ is a σ -finite measure space. Our example arises from the functional central limit theorem for a sequence of $L_2(\mu)$ -valued random elements.

Let $\{X_k\} = \{X_k : k \geq 1\}$ be random elements with values in the separable Banach space \mathbb{E} given by

$$X_k = \sum_{j=0}^{\infty} u_j \varepsilon_{k-j} \quad (1)$$

for each $k \geq 1$, where $\{u_j\} = \{u_j : j \geq 0\} \subset L(\mathbb{E})$ and $\{\varepsilon_k\} = \{\varepsilon_k : k \in \mathbb{Z}\}$ are independent and identically distributed \mathbb{E} -valued random elements with $\mathbb{E} \varepsilon_0 = 0$, $\mathbb{E} \|\varepsilon_0\|^2 < \infty$, where $\|\cdot\|$ is the norm of the Banach space \mathbb{E} . Let $\{\zeta_n\} = \{\zeta_n(t) : t \in [0, 1]\}_{n \geq 1}$ be random polygonal functions (piecewise linear functions) constructed from the partial sums $\{S_n\} = \{S_n = X_1 + \dots + X_n : n \geq 1\}$. The asymptotic behaviour of $\{S_n\}$ and that of $\{\zeta_n\}$ strongly depend on the convergence of the series $\sum_{j=0}^{\infty} \|u_j\|$, where $\|\cdot\|$ is the operator norm. Roughly speaking, if the series $\sum_{j=0}^{\infty} \|u_j\|$ converges, the asymptotic behaviour of $\{S_n\}$ and $\{\zeta_n\}$ is inherited from $\{\varepsilon_k\}$ (see Merlevède, Peligrad, and Utev [17], Račkauskas and Suquet [18] for more details). However, this is not the case when $\sum_{j=0}^{\infty} \|u_j\| = \infty$ (see Račkauskas and Suquet [19] and Characiejus and Račkauskas [2]).

Račkauskas and Suquet [19] consider $\{X_k\}$ with values in an abstract separable Hilbert space \mathbb{H} when $\sum_{j=0}^{\infty} \|u_j\| = \infty$ with $u_0 = I$ and $u_j = j^{-T}$ for $j \geq 1$, where $T \in L(\mathbb{H})$ satisfies $\frac{1}{2}I < T < I$ and T commutes with the covariance operator of ε_0 . We obtain an operator self-similar process with the covariance structure different from Račkauskas and Suquet [19] since T does not necessarily commute with the covariance operator of ε_0 in our case.

Specifically, we investigate $\{X_k\}$ with values in $L_2(\mu)$ and $\{u_j\}$ given by

$$u_j = (j+1)^{-D} \quad (2)$$

for each $j \geq 0$, where D is a multiplication operator defined by $Df = \{d(s)f(s) : s \in \mathbb{S}\}$ for each $f \in L_2(\mu)$ with a measurable function $d : \mathbb{S} \rightarrow \mathbb{R}$. Our main results (Theorem 1 and Theorem 2 in Section 4) establish sufficient conditions for the convergence in distribution of ζ_n in the space $C([0, 1]; L_2(\mu))$ in the following two cases: either $d \in (1/2, 1)$ (shorthand for $1/2 < d(s) < 1$ for all $s \in \mathbb{S}$) or $d = 1$ (shorthand for $d(s) = 1$ for all $s \in \mathbb{S}$). In the former case, we provide sufficient conditions for the convergence in distribution of $n^{-H}\zeta_n$ to a Gaussian stochastic process \mathcal{G} , where $\{n^{-H}\}$ are multiplication operators given by $n^{-H}f = \{n^{-[3/2-d(s)]}f(s) : s \in \mathbb{S}\}$ for each $n \geq 1$ and $f \in L_2(\mu)$. In the latter case, we establish convergence in distribution of $(\sqrt{n} \log n)^{-1}\zeta_n$ to an $L_2(\mu)$ -valued Wiener process. The results of this paper generalize our previous results since in Characiejus and Račkauskas [2] only the central limit theorem is investigated.

The rest of the paper is organized as follows. In Section 2, we give two alternative ways to construct $\{X_k\}$ and establish some properties of $\{X_k\}$ and $\{\zeta_n\}$. The existence of an operator self-similar process \mathcal{X} with values in $L_2(\mu)$ is established in Section 3. In Section 4, we establish sufficient conditions for the functional central limit theorem.

2 Preliminaries

2.1 Construction of $\{X_k\}$

There are two approaches to construct $\{X_k\}$ with values in $L_2(\mu)$. The first approach is to define $\{X_k\}$ as stochastic processes with space varying memory and square μ -integrable sample paths. The second approach is to define $L_2(\mu)$ valued random variable X_k for each $k \geq 1$ as series (1) with u_j given by (2) and to investigate the convergence of such series. We present both of these two approaches.

First approach

Let $\{\varepsilon_k\} = \{\varepsilon_k(s) : s \in \mathbb{S}\}_{k \in \mathbb{Z}}$ be independent and identically distributed measurable stochastic processes defined on the probability space (Ω, \mathcal{F}, P) , i.e. $\{\varepsilon_k\}$ are $\mathcal{F} \otimes \mathcal{S}$ -measurable functions $\varepsilon_k : \Omega \times \mathbb{S} \rightarrow \mathbb{R}$. We require that $E\varepsilon_0(s) = 0$ and $E\varepsilon_0^2(s) < \infty$ for each $s \in \mathbb{S}$ and denote

$$\sigma(r, s) = E[\varepsilon_0(r)\varepsilon_0(s)], \quad \sigma^2(s) = E\varepsilon_0^2(s), \quad r, s \in \mathbb{S}.$$

Define stochastic processes $\{X_k\} = \{X_k(s) : s \in \mathbb{S}\}_{k \geq 1}$ by setting

$$X_k(s) = \sum_{j=0}^{\infty} (j+1)^{-d(s)} \varepsilon_{k-j}(s) \quad (3)$$

for each $s \in \mathbb{S}$ and each $k \geq 1$. Observe that $d(s) > 1/2$ is a necessary and sufficient condition for the almost sure convergence of series (3) (this fact follows from Kolmogorov's three-series theorem). It is well-known that the growth rate of the partial sums $\{\sum_{k=1}^n X_k(s)\}$ depends on $d(s)$. Viewing \mathbb{S} as the set of space indexes and \mathbb{Z} as the set of time indexes, we thus have a functional process $\{X_k\}$ with space varying memory. We refer to Giraitis, Koul, and Surgailis [8] for an encyclopedic treatment of long memory phenomenon of stochastic processes.

We denote

$$\gamma_h(r, s) = E[X_0(r)X_h(s)], \quad \gamma_h(s) = E[X_0(s)X_h(s)], \quad r, s \in \mathbb{S}, \quad h \in \mathbb{N}.$$

For fixed $r, s \in \mathbb{S}$, the sequences $\{X_k(r)\}$ and $\{X_k(s)\}$ are stationary sequences of random variables with zero means and cross-covariance

$$\gamma_h(r, s) = \sigma(r, s) \sum_{j=0}^{\infty} (j+1)^{-d(r)} (j+h+1)^{-d(s)}. \quad (4)$$

Throughout the paper

$$d(r, s) = d(r) + d(s), \quad r, s \in \mathbb{S}, \quad (5)$$

and

$$c(r, s) = \int_0^{\infty} x^{-d(r)} (x+1)^{-d(s)} dx, \quad r, s \in \mathbb{S}, \quad (6)$$

provided that $1/2 < d(r) < 1$, $d(s) > 1/2$. Let us observe that $c(r, s) = B(1 - d(r), d(r, s) - 1)$, where B is the beta function. If $r = s$, we denote $c(r, s)$ by $c(s)$. $c(s)$ can be estimated from above with the following inequality

$$c(s) \leq \frac{1}{1 - d(s)} + \frac{1}{2d(s) - 1}. \quad (7)$$

Proposition 1 gives the asymptotic behaviour of $\gamma_h(r, s)$ and Proposition 2 provides a necessary and sufficient condition for the summability of the series $\sum_{k=0}^{\infty} \gamma_k(r, s)$ (for the proof, see Characiejus and Račkauskas [2]). The notation $a_n \sim b_n$ indicates that the ratio of the two sequences tends to 1 as $n \rightarrow \infty$.

Proposition 1. *If $1/2 < d(r) < 1$ and $d(s) > 1/2$, then*

$$\gamma_h(r, s) \sim c(r, s) \sigma(r, s) \cdot h^{1-d(r,s)}.$$

If $d(r) = d(s) = 1$, then

$$\gamma_h(r, s) \sim \sigma(r, s) \cdot h^{-1} \log h.$$

Proposition 2. *The series*

$$\sum_{k=0}^{\infty} \gamma_k(r, s)$$

converges if and only if $d(r) > 1$ and $d(r, s) > 2$.

Remark 1. The series $\sum_{k=0}^{\infty} \gamma_k(s)$ converges if and only if $d(s) > 1$.

Let $\mathcal{L}_2(\mu) = \mathcal{L}_2(\mathbb{S}, \mathcal{S}, \mu)$ be a separable space of real valued square μ -integrable functions with a seminorm

$$\|f\| = \left[\int_{\mathbb{S}} |f(v)|^2 \mu(dv) \right]^{1/2}, \quad f \in \mathcal{L}_2(\mu),$$

and let $L_2(\mu) = L_2(\mathbb{S}, \mathcal{S}, \mu)$ be the corresponding Hilbert space of equivalence classes of μ -almost everywhere equal functions with an inner product

$$\langle f, g \rangle = \int_{\mathbb{S}} f(v)g(v) \mu(dv), \quad f, g \in L_2(\mu).$$

With an abuse of notation, we denote by f both a function and its equivalence class to avoid cumbersome notation. The intended meaning should be clear from the context.

Proposition 3 establishes a necessary and sufficient condition for the sample paths of the stochastic process $\{X_k(s) : s \in \mathbb{S}\}$ to be almost surely square μ -integrable with $E \|X_k\|^2 < \infty$ for each $k \geq 1$ (see Characiejus and Račkauskas [2] for the proof).

Proposition 3. *The sample paths of the stochastic process $\{X_k(s) : s \in \mathbb{S}\}$ almost surely belong to the space $\mathcal{L}_2(\mu)$ and $E \|X_k\|^2 < \infty$ for each $k \in \mathbb{Z}$ if and only if both of the integrals*

$$E \|\varepsilon_0\|^2 = \int_{\mathbb{S}} \sigma^2(v) \mu(dv) \quad \text{and} \quad \int_{\mathbb{S}} \frac{\sigma^2(v)}{2d(v) - 1} \mu(dv)$$

are finite.

A stochastic process $\{\xi(s) : s \in \mathbb{S}\}$ defined on a probability space (Ω, \mathcal{F}, P) with sample paths in $\mathcal{L}_2(\mu)$ induces the $\mathcal{F} - \mathcal{B}(L_2(\mu))$ -measurable function $\omega \rightarrow \{\xi(s)(\omega) : s \in \mathbb{S}\} : \Omega \rightarrow L_2(\mu)$, where $\mathcal{B}(L_2(\mu))$ is the Borel σ -algebra of $L_2(\mu)$ (for more details, see Cremers and Kadelka [3]). Therefore we shall frequently consider each stochastic process $\{\xi(s) : s \in \mathbb{S}\}$ with sample paths in $\mathcal{L}_2(\mu)$ as a random element with values in $L_2(\mu)$ and denote it by $\{\xi(s) : s \in \mathbb{S}\}$ or simply by ξ .

Second approach

Now we establish a necessary and sufficient condition for the mean square convergence of series (1) with $\{u_j\}$ given by (2). Recall that $(j+1)^{-D}f = \{(j+1)^{-d(s)}f(s) : s \in \mathbb{S}\}$ for each $j \geq 0$ and $f \in L_2(\mu)$ since $e^T = \sum_{j=0}^{\infty} T^j/j!$ and $\lambda^T = e^{T \log \lambda}$ for $T \in L(\mathbb{E})$ and $\lambda > 0$.

Proposition 4. *Series (1) with u_j given by (2) and $L_2(\mu)$ -valued random elements $\{\varepsilon_k\}$ such that $E\varepsilon_0 = 0$ and $E\|\varepsilon_0\|^2 < \infty$ converges in mean square if and only if there exists a measurable set $\mathbb{S}_0 \subset \mathbb{S}$ such that $\mu(\mathbb{S} \setminus \mathbb{S}_0) = 0$, $d(s) > 1/2$ for all $s \in \mathbb{S}_0$ and the integral*

$$\int_{\mathbb{S}} \frac{\sigma^2(v)}{2d(v) - 1} \mu(dv)$$

is finite.

Proof. Let $N > M$, $\sigma^2(s) = E\varepsilon_0^2(s)$, $s \in \mathbb{S}$, and observe that

$$E \left\| \sum_{j=M+1}^N u_j \varepsilon_{j-k} \right\|^2 = \sum_{j=M+1}^N \int_{\mathbb{S}} (j+1)^{-2d(v)} \sigma^2(v) \mu(dv).$$

Since

$$\sum_{j=0}^{\infty} \int_{\mathbb{S}} (j+1)^{-2d(r)} \sigma^2(r) \mu(dr) = \int_{\mathbb{S}} \sum_{j=1}^{\infty} j^{-2d(r)} \sigma^2(r) \mu(dr)$$

and

$$\frac{1}{2d(r) - 1} \leq \sum_{j=1}^{\infty} j^{-2d(r)} \leq 1 + \frac{1}{2d(r) - 1}$$

we have that

$$\int_{\mathbb{S}} \frac{\sigma^2(r)}{2d(r) - 1} \mu(dr) \leq \int_{\mathbb{S}} \sigma^2(r) \sum_{j=1}^{\infty} j^{-2d(r)} \mu(dr) \leq E\|\varepsilon_0\|^2 + \int_{\mathbb{S}} \frac{\sigma^2(r)}{2d(r) - 1} \mu(dr)$$

and the proof is complete. \square

Remark 2. Since $\{\varepsilon_k\}$ are independent, it follows from Lévy-Itô-Nisio theorem (see Ledoux and Talagrand [14], Theorem 6.1, p. 151) and Proposition 4 that series (1) also converges almost surely. Hence, X_k for each $k \geq 1$ is an $L_2(\mu)$ -valued random element and Proposition 4 is consistent with Proposition 3.

Remark 3. Since u_j given by (2) are multiplication operators from $L_2(\mu)$ to $L_2(\mu)$, we have that the operator norm $\|u_j\| = \inf\{c > 0 : \mu(s \in \mathbb{S} : |(j+1)^{-d(s)}| > c) = 0\}$. If $\underline{d} = \text{ess inf } d = 1/2$, then we have that $\sum_{j=0}^{\infty} \|u_j\|^2 = \sum_{j=1}^{\infty} j^{-1} = \infty$, but series (1) might still converge. The square summability of the operator norms of u_j is not a necessary condition for the almost sure convergence of series (1).

2.2 Random polygonal functions $\{\zeta_n\}$

Let $\{\zeta_n\} = \{\zeta_n(t) : t \in [0, 1]\}_{n \geq 1}$ be random polygonal functions (piecewise linear functions) constructed from partial sums $S_k = X_1 + \dots + X_k$, $k \geq 1$:

$$\zeta_n(t) = S_{[nt]} + \{nt\}X_{[nt]+1}$$

for each $n \geq 1$ and each $t \in [0, 1]$, where $\lfloor \cdot \rfloor$ is the floor function defined by $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$ for $x \in \mathbb{R}$ and $\{x\} = x - \lfloor x \rfloor$ is the fractional part of $x \in \mathbb{R}$. We adopt the usual convention that an empty sum equals 0. For each $t \in [0, 1]$ the random function $\zeta_n(t)$ can be expressed as a series

$$\zeta_n(t) = \sum_{j=-\infty}^{\lfloor nt \rfloor + 1} a_{nj}(t) \varepsilon_j,$$

where

$$a_{nj}(t) = \sum_{k=1}^{\lfloor nt \rfloor} v_{k-j} + \{nt\} v_{\lfloor nt \rfloor + 1 - j} \quad (8)$$

and

$$v_j = \begin{cases} u_j, & \text{if } j \geq 0; \\ 0, & \text{if } j < 0. \end{cases} \quad (9)$$

Denote $\zeta_n(s, t) = \sum_{k=1}^{\lfloor nt \rfloor} X_k(s) + \{nt\} X_{\lfloor nt \rfloor + 1}(s)$ for $s \in \mathbb{S}$ and $t \in [0, 1]$. Each random variable $\zeta_n(s, t)$ can be expressed as a series $\zeta_n(s, t) = \sum_{j=-\infty}^{\lfloor nt \rfloor + 1} a_{nj}(s, t) \varepsilon_j(s)$, where

$$a_{nj}(s, t) = \sum_{k=1}^{\lfloor nt \rfloor} v_{k-j}(s) + \{nt\} v_{\lfloor nt \rfloor + 1 - j}(s) \quad (10)$$

and

$$v_j(s) = \begin{cases} (j+1)^{-d(s)}, & \text{if } j \geq 0; \\ 0, & \text{if } j < 0. \end{cases} \quad (11)$$

Observe that $v_j = v_j(s)$ if $d = 1$ since $u_j = (j+1)^{-1}$ if $d = 1$. Notice that the upper bounds of summation of the series in the expressions of $\zeta_n(t)$ and $\zeta_n(s, t)$ can be extended up to ∞ since $a_{nj}(s, t) = 0$ and $a_{nj}(t) = 0$ if $j > \lfloor nt \rfloor + 1$.

Set $\mathbb{T} = \mathbb{S} \times [0, \infty)$ and define the function $V : \mathbb{T}^2 \rightarrow \mathbb{R}$ by

$$\begin{aligned} V((r, t), (s, u)) &= \\ &= \frac{\sigma(r, s)}{[2 - d(r, s)][3 - d(r, s)]} [c(s, r)t^{3-d(r, s)} + c(r, s)u^{3-d(r, s)} - C(r, s; t - u)|t - u|^{3-d(r, s)}], \end{aligned} \quad (12)$$

where $d(r, s)$ is given by (5), $c(r, s)$ is given by (6) and

$$C(r, s; t) = \begin{cases} c(r, s) & \text{if } t < 0; \\ c(s, r) & \text{if } t > 0. \end{cases}$$

Now we are prepared to derive the asymptotic behavior of the sequence of cross-covariances of ζ_n .

Proposition 5. *Suppose either $1/2 < d(r) < 1$ and $1/2 < d(s) < 1$ or $d(r) = d(s) = 1$. In both cases, the following asymptotic relation holds*

$$\mathbb{E}[\zeta_n(r, t)\zeta_n(s, u)] \sim \mathbb{E}[S_{\lfloor nt \rfloor}(r)S_{\lfloor nu \rfloor}(s)].$$

Proposition 6. *If $1/2 < d(r) < 1$ and $1/2 < d(s) < 1$, then*

$$\mathbb{E}[S_{[nt]}(r)S_{[nu]}(s)] \sim V((r, t), (s, u)) \cdot n^{3-d(r,s)}$$

for $(r, t), (s, u) \in \mathbb{S} \times [0, 1]$, where V is given by (12).

If $d(r) = d(s) = 1$, then

$$\mathbb{E}[S_{[nt]}(r)S_{[nu]}(s)] \sim \sigma(r, s) \cdot \min(t, u) \cdot n \log^2 n.$$

Remark 4. Let us assume that $r = s$ and $1/2 < d(s) < 1$. By setting $r = s$ in Proposition 6 and using Proposition 5, we obtain that

$$\mathbb{E}[\zeta_n(s, t)\zeta_n(s, u)] \sim \frac{\sigma^2(s)c(s)}{[1 - d(s)][3 - 2d(s)]} \cdot \mathbb{E}[B_{3/2-d(s)}(t)B_{3/2-d(s)}(u)] \cdot n^{3-2d(s)},$$

where

$$\mathbb{E}[B_{3/2-d(s)}(t)B_{3/2-d(s)}(u)] = \frac{1}{2}[t^{3-2d(s)} + u^{3-2d(s)} - |t - u|^{3-2d(s)}]$$

is the covariance function of the fractional Brownian motion $B_{3/2-d(s)} = \{B_{3/2-d(s)}(t) : t \in [0, 1]\}$ with the Hurst parameter $3/2 - d(s)$ and $c(s) = c(s, s)$ is given by (6).

Remark 5. The asymptotic behaviour of the variance $\mathbb{E}\zeta_n^2(s, t)$ follows from Proposition 5 and Proposition 6 by setting $r = s$ and $t = u$: if $1/2 < d(s) < 1$, then

$$\mathbb{E}\zeta_n^2(s, t) \sim \frac{c(s)\sigma^2(s)}{[1 - d(s)][3 - 2d(s)]} \cdot t^{3-2d(s)} \cdot n^{3-2d(s)};$$

if $d(s) = 1$, then

$$\mathbb{E}\zeta_n^2(s, t) \sim \sigma^2(s) \cdot t \cdot n \log^2 n.$$

Proof of Proposition 6. Suppose $t < u$ and split the cross-covariance of the partial sums into two terms

$$\mathbb{E}[S_{[nt]}(r)S_{[nu]}(s)] = \mathbb{E}[S_{[nt]}(r)S_{[nt]}(s)] + \mathbb{E}[S_{[nt]}(r)[S_{[nu]}(s) - S_{[nt]}(s)]]. \quad (13)$$

The following two asymptotic relations are proved in Characiejus and Račkauskas [2]: if $1/2 < d(r) < 1$ and $1/2 < d(s) < 1$, then

$$\mathbb{E}[S_n(r)S_n(s)] \sim \frac{[c(r, s) + c(s, r)]\sigma(r, s)}{[2 - d(r, s)][3 - d(r, s)]} \cdot n^{3-d(r,s)}; \quad (14)$$

if $d(r) = d(s) = 1$, then

$$\mathbb{E}[S_n(r)S_n(s)] \sim \sigma(r, s) \cdot n \log^2 n. \quad (15)$$

The asymptotic behaviour of the first term of sum (13) is established using (14) and (15): if $1/2 < d(r) < 1$ and $1/2 < d(s) < 1$, then

$$\mathbb{E}[S_{[nt]}(r)S_{[nt]}(s)] \sim \frac{[c(r, s) + c(s, r)]\sigma(r, s)}{[2 - d(r, s)][3 - d(r, s)]} \cdot t^{3-d(r,s)} \cdot n^{3-d(r,s)}; \quad (16)$$

if $d(r) = d(s) = 1$, then

$$\mathbb{E}[S_{[nt]}(r)S_{[nt]}(s)] \sim \sigma(r, s) \cdot t \cdot n \log^2 n. \quad (17)$$

In order to establish the asymptotic behaviour of the second term of sum (13), we express it in the following way

$$\begin{aligned} \mathbb{E}[S_{[nt]}(r)[S_{[nu]}(s) - S_{[nt]}(s)]] &= \sum_{k=1}^{m_n-1} k[\gamma_k(r, s) + \gamma_{[nu]-k}(r, s)] \\ &\quad + m_n \sum_{k=0}^{|[nu]-2[nt]|} \gamma_{m_n+k}(r, s), \end{aligned} \quad (18)$$

where $m_n := \min([nt], [nu] - [nt])$ (we also use the notation $m := \min(t, u - t)$). For simplicity, denote

$$\kappa(a, b) = \sum_{k=a+1}^b \gamma_k(r, s) \quad \text{and} \quad \nu(a, b) = \sum_{k=a+1}^b k\gamma_k(r, s).$$

Then we have that

$$\sum_{k=1}^{m_n-1} k\gamma_{[nu]-k}(r, s) = [nu]\kappa([nu] - m_n, [nu] - 1) - \nu([nu] - m_n, [nu] - 1). \quad (19)$$

Let us recall a few facts about sequences. We use these facts to establish asymptotic behaviour of the sums in (18) and (19). Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of positive real numbers such that $a_n \sim b_n$. Then $\sum_{k=1}^n a_k \sim \sum_{k=1}^n b_k$ provided either of these partial sums diverges. Let f be a continuous strictly increasing or strictly decreasing function such that $f(x)/f(x+1) \rightarrow 1$ as $x \rightarrow \infty$ and $\int_1^n f(x)dx \rightarrow \infty$ as $n \rightarrow \infty$. Then $\sum_{k=1}^n f(k) \sim \int_1^n f(x)dx$.

Since $\gamma_k(r, s) \sim c(r, s)\sigma(r, s) \cdot k^{1-d(r,s)}$ if $1/2 < d(r) < 1$ and $d(s) > 1/2$ (see Proposition 1), we obtain the following asymptotic relations using the facts about sequences mentioned above:

$$\nu(0, m_n - 1) \sim \frac{c(r, s)\sigma(r, s)m_n^{3-d(r,s)}}{3-d(r, s)} \cdot n^{3-d(r,s)}; \quad (20)$$

$$[nu]\kappa([nu] - m_n, [nu] - 1) \sim \frac{c(r, s)\sigma(r, s)u[u^{2-d(r,s)} - (u - m)^{2-d(r,s)}]}{2-d(r, s)} \cdot n^{3-d(r,s)}; \quad (21)$$

$$\nu([nu] - m_n, [nu] - 1) \sim \frac{c(r, s)\sigma(r, s)[u^{3-d(r,s)} - (u - m)^{3-d(r,s)}]}{3-d(r, s)} \cdot n^{3-d(r,s)}; \quad (22)$$

$$\begin{aligned} m_n\kappa(m_n - 1, m_n + |[nu] - 2[nt]|) &\sim \\ &\sim \frac{c(r, s)\sigma(r, s)m_n[(m + |u - 2t|)^{2-d(r,s)} - m^{2-d(r,s)}]}{2-d(r, s)} \cdot n^{3-d(r,s)}. \end{aligned} \quad (23)$$

We have that

$$\begin{aligned} \mathbb{E}[S_{[nt]}(r)[S_{[nu]}(s) - S_{[nt]}(s)]] &\sim \\ &\sim \frac{c(r, s)\sigma(r, s)}{[2-d(r, s)][3-d(r, s)]} [-t^{3-d(r,s)} + u^{3-d(r,s)} - (u - t)^{3-d(r,s)}] \cdot n^{3-d(r,s)} \end{aligned} \quad (24)$$

using asymptotic relations (20)-(23). Combining (16) with (24), we obtain

$$\begin{aligned} \mathbb{E}[S_{[nt]}(r)S_{[nu]}(s)] &\sim \\ &\sim \frac{\sigma(r, s)}{[2-d(r, s)][3-d(r, s)]} [c(r, s)t^{3-d(r,s)} + c(r, s)[u^{3-d(r,s)} - (u - t)^{3-d(r,s)}]] \cdot n^{3-d(r,s)}. \end{aligned}$$

Similarly, if $d(r) = d(s) = 1$, then $\gamma_k(r, s) \sim \sigma(r, s) \cdot k^{-1} \log k$ (see Proposition 1) and the following asymptotic relations are true

$$\nu(0, m_n - 1) \sim \sigma(r, s) m \cdot n \log n; \quad (25)$$

$$\lfloor nu \rfloor \kappa(\lfloor nu \rfloor - m_n, \lfloor nu \rfloor - 1) \sim \sigma(r, s) [\log u - \log(u - m)] u \cdot n \log n; \quad (26)$$

$$\nu(\lfloor nu \rfloor - m_n, \lfloor nu \rfloor - 1) \sim \sigma(r, s) m \cdot n \log n; \quad (27)$$

$$m_n \kappa(m_n - 1, m_n + |\lfloor nu \rfloor - 2\lfloor nt \rfloor|) \sim \sigma(r, s) [\log(m + |u - 2t|) - \log m] m \cdot n \log n. \quad (28)$$

Since sequences (25)-(28) grow slower than sequence (17), we conclude that

$$\mathbb{E}[S_{\lfloor nt \rfloor}(r) S_{\lfloor nu \rfloor}(s)] \sim \sigma(r, s) \cdot t \cdot n \log^2 n.$$

If $t > u$, the proof is exactly the same as in the case of $t < u$. If $t = u$, then we just use asymptotic relations (16) and (17). The proof of Proposition 6 is complete. \square

Proof of Proposition 5. We have that

$$\begin{aligned} \mathbb{E}[\zeta_n(r, t) \zeta_n(s, u)] &= \mathbb{E}[S_{\lfloor nt \rfloor}(r) S_{\lfloor nu \rfloor}(s)] \\ &\quad + \{nu\} \mathbb{E}[S_{\lfloor nt \rfloor}(r) X_{\lfloor nu \rfloor + 1}(s)] \\ &\quad + \{nt\} \mathbb{E}[S_{\lfloor nu \rfloor}(s) X_{\lfloor nt \rfloor + 1}(r)] \\ &\quad + \{nt\} \{nu\} \mathbb{E}[X_{\lfloor nt \rfloor + 1}(r) X_{\lfloor nu \rfloor + 1}(s)] \end{aligned}$$

and

$$\mathbb{E}[S_{\lfloor nt \rfloor}(r) X_{\lfloor nu \rfloor + 1}(s)] \leq \lfloor nt \rfloor \gamma_0(r, s).$$

The result follows from Proposition 6 since $\mathbb{E}[S_{\lfloor nt \rfloor}(r) S_{\lfloor nu \rfloor}(s)]$ is the only term in the expression of $\mathbb{E}[\zeta_n(r, t) \zeta_n(s, u)]$ that grows faster than linearly. \square

3 Operator self-similar process

In this section, we show that there exists a Gaussian stochastic process $\mathcal{X} = \{\mathcal{X}(s, t) : (s, t) \in \mathbb{T}\}$ with zero mean and covariance function V given by (12). The stochastic process $\{\mathcal{X}(\cdot, t) : t \in [0, \infty)\}$ is an operator self-similar process with values in $L_2(\mu)$.

We begin by showing that the function V is a covariance function.

Proposition 7. *The function $V : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$, given by (12), with $d \in (1/2, 1)$ is a covariance function of a stochastic process indexed by the set \mathbb{T} .*

Proof. It follows from equation (12) that the function V is symmetric, i.e.

$$V(\tau, \tau') = V(\tau', \tau), \quad \tau, \tau' \in \mathbb{T}.$$

So we need to prove that the function V is positive definite. Let $N \in \mathbb{N}$, $\tau_i = (s_i, t_i) \in \mathbb{T}$ and $w_i \in \mathbb{R}$, where $i \in \{1, \dots, N\}$. Denote $M = \max\{t_1, \dots, t_N\}$ and $\tilde{w}_i = w_i M^{3/2-d(s_i)}$, $i \in \{1, \dots, N\}$. Using equation (12) and Propositions 5 and 6, we obtain that

$$\begin{aligned} \sum_{i=1}^N \sum_{j=1}^N w_i w_j V(\tau_i, \tau_j) &= \sum_{i=1}^N \sum_{j=1}^N w_i w_j M^{3-[d(s_i)+d(s_j)]} V((s_i, t_i/M), (s_j, t_j/M)) \\ &= \sum_{i=1}^N \sum_{j=1}^N \tilde{w}_i \tilde{w}_j \lim_{n \rightarrow \infty} \frac{1}{n^{3-[d(s_i)+d(s_j)]}} \mathbb{E}[\zeta_n(s_i, t_i/M) \zeta_n(s_j, t_j/M)] \geq 0 \end{aligned}$$

since

$$\frac{1}{n^{3-d(r,s)}} \mathbb{E}[\zeta_n(r, t) \zeta_n(s, u)]$$

is a covariance function for all $(r, t), (s, u) \in \mathbb{S} \times [0, 1]$ and for all $n \in \mathbb{N}$. \square

Let us recall that a random element ξ with values in a separable Banach space \mathbb{E} is Gaussian if for any continuous linear functional f on \mathbb{E} , $f(\xi)$ is real valued Gaussian random variable. A stochastic process $\{\xi_t : t \in T\}$ with values in \mathbb{E} is Gaussian if each finite linear combination $\sum_i \alpha_i \xi_{t_i}$, $\alpha_i \in \mathbb{R}$, $t_i \in T$, is Gaussian random element in \mathbb{E} (for more details about Gaussian random elements and Gaussian stochastic processes with values in Banach spaces, see the textbook by Ledoux and Talagrand [14]).

We have the following corollary of Proposition 7.

Corollary. *There exists a zero mean Gaussian stochastic process $\mathcal{X} = \{\mathcal{X}(s, t) : (s, t) \in \mathbb{T}\}$ with the covariance function V given by (12).*

Next we describe the sample path properties of the stochastic process \mathcal{X} . First we consider for each $t \in [0, \infty)$ the stochastic process $\{\mathcal{X}(s, t) : s \in \mathbb{S}\}$.

Proposition 8. *If $d \in (1/2, 1)$ and the integrals*

$$\int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)]^2} \mu(dv) \quad \text{and} \quad \int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)][2d(v)-1]} \mu(dv)$$

are finite, then for each $t \in [0, \infty)$ the stochastic process $\{\mathcal{X}(s, t) : s \in \mathbb{S}\}$ has sample paths in $\mathcal{L}_2(\mu)$ and induces a Gaussian random element with values in $L_2(\mu)$ which is denoted by $\mathcal{X}(\cdot, t)$. Moreover, the process $\{\mathcal{X}(\cdot, t) : t \in [0, \infty)\}$ with values in $L_2(\mu)$ is Gaussian.

Proof. Since we have that

$$\begin{aligned} \mathbb{E} \int_{\mathbb{S}} \mathcal{X}^2(v, t) \mu(dv) &= \int_{\mathbb{S}} \frac{\sigma^2(v) c(v)}{[1-d(v)][3-2d(v)]} \cdot t^{3-2d(v)} \mu(dv) \\ &\leq \max\{t, t^2\} \left[\int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)]^2} \mu(dv) + \int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)][2d(v)-1]} \mu(dv) \right] \end{aligned}$$

using inequality (7) to estimate $c(s)$ from above, the sample paths of the stochastic process $\{\mathcal{X}(s, t) : s \in \mathbb{S}\}$ almost surely belong to the space $\mathcal{L}_2(\mu)$ for each $t \in [0, \infty)$. Hence $\mathcal{X}(\cdot, t)$ is a random element in $L_2(\mu)$. Clearly it is a Gaussian one. \square

Finally, we show that the stochastic process $\{\mathcal{X}(\cdot, t) : t \in [0, \infty)\}$ is operator self-similar.

Proposition 9. *The stochastic process $\{\mathcal{X}(\cdot, t) : t \in [0, \infty)\}$ is operator self-similar with scaling family of operators $\{a^H : a > 0\}$ where a^H , $a > 0$, is a multiplication operator defined by $a^H f = \{a^{3/2-d(s)} f(s) : s \in \mathbb{S}\}$ for $f \in L_2(\mu)$.*

Proof. We need to show that

$$\{\mathcal{X}(\cdot, at) : t \in [0, \infty)\} \stackrel{fdd}{=} \{a^H \mathcal{X}(\cdot, t) : t \in [0, \infty)\}. \quad (29)$$

Since stochastic processes on both sides of equality (29) are zero-mean Gaussian stochastic processes, we only need to show that their covariance structure is the same. Using the fact that

two operators A and B are equal if and only if $\langle Af, g \rangle = \langle Bf, g \rangle$ for all $f, g \in L_2(\mu)$ and the fact that

$$E[a^{3/2-d(r)}\mathcal{X}(r, t)a^{3/2-d(s)}\mathcal{X}(s, u)] = E[\mathcal{X}(r, at)\mathcal{X}(s, au)] \quad (30)$$

for all $r, s \in \mathbb{S}$ and $t, u \in [0, \infty)$ (equality (30) follows from equation (12)), we conclude the proof by showing that

$$\begin{aligned} \langle E[\langle a^H \mathcal{X}(\cdot, t), f \rangle a^H \mathcal{X}(\cdot, u)], g \rangle &= \int_{\mathbb{S}} E \left[\left(\int_{\mathbb{S}} a^{3/2-d(u)} \mathcal{X}(u, t) f(u) \mu(du) \right) a^{3/2-d(r)} \mathcal{X}(r, u) \right] g(r) \mu(dr) \\ &= \int_{\mathbb{S}} \left(\int_{\mathbb{S}} E[a^{3/2-d(u)} \mathcal{X}(u, t) a^{3/2-d(r)} \mathcal{X}(r, u)] f(u) \mu(du) \right) g(r) \mu(dr) \\ &= \langle E[\langle \mathcal{X}(\cdot, at), f \rangle \mathcal{X}(\cdot, au)], g \rangle \end{aligned}$$

for all $f, g \in L_2(\mu)$. □

4 Main results

4.1 Functional central limit theorem

We shall consider $\{\zeta_n\}$ as random elements in a separable Banach space $C([0, 1]; L_2(\mu))$ of continuous functions $f : [0, 1] \rightarrow L_2(\mu)$ endowed with the norm

$$\|f\| = \sup_{t \in [0, 1]} \left[\int_{\mathbb{S}} f^2(v, t) \mu(dv) \right]^{1/2}, \quad f \in C([0, 1]; L_2(\mu)).$$

Before stating sufficient conditions for the functional central limit theorem, we define the limit Gaussian processes

$$\mathcal{G} = \{\mathcal{G}(s, t) : (s, t) \in \mathbb{S} \times [0, 1]\} \quad \text{and} \quad \mathcal{G}' = \{\mathcal{G}'(s, t) : (s, t) \in \mathbb{S} \times [0, 1]\}.$$

Let the stochastic process \mathcal{G} be a restriction to $\mathbb{S} \times [0, 1]$ of the stochastic process $\mathcal{X} = \{\mathcal{X}(s, t) : (s, t) \in \mathbb{S} \times [0, \infty)\}$ defined in Section 3. Let the stochastic process \mathcal{G}' be Gaussian with the covariance function $E[\mathcal{G}'(r, t)\mathcal{G}'(s, u)] = \sigma(r, s) \min(t, u)$, $(r, t), (s, u) \in \mathbb{S} \times [0, 1]$. If the integral $\int_{\mathbb{S}} \sigma^2(v) \mu(dv)$ is finite, then for each $t \in [0, 1]$ the sample paths of the stochastic process $\{\mathcal{G}'(s, t) : s \in \mathbb{S}\}$ belong to the space $\mathcal{L}_2(\mu)$ (the proof is basically the same as the proof of Proposition 8).

The following proposition establishes conditions under which both of the stochastic processes $\{\mathcal{G}(\cdot, t) : t \in [0, 1]\}$ and $\{\mathcal{G}'(\cdot, t) : t \in [0, 1]\}$ with values in the space $L_2(\mu)$ have continuous versions.

Proposition 10. *If the integrals*

$$\int_{\mathbb{S}} \frac{\sigma^2(v)}{[1 - d(v)]^2} \mu(dv) \quad \text{and} \quad \int_{\mathbb{S}} \frac{\sigma^2(v)}{[1 - d(v)][2d(v) - 1]} \mu(dv)$$

are finite, then the $L_2(\mu)$ -valued stochastic process $\{\mathcal{G}(\cdot, t) : t \in [0, 1]\}$ has a continuous version.

If the integral

$$\int_{\mathbb{S}} \sigma^2(v) \mu(dv)$$

is finite, then the $L_2(\mu)$ -valued stochastic process $\{\mathcal{G}'(\cdot, t) : t \in [0, 1]\}$ has a continuous version.

Proof. We use the following inequality for the moments of a Gaussian random element ξ with values in a separable Banach space:

$$(\mathbb{E} \|\xi\|^p)^{1/p} \leq K_{p,q} (\mathbb{E} \|\xi\|^q)^{1/q}, \quad (31)$$

where $0 < p, q < \infty$ and $K_{p,q}$ is a constant depending on p and q only (for the proof, see Ledoux and Talagrand [14], p. 59, Corollary 3.2).

Using Kolmogorov's continuity theorem (see the textbook by Kallenberg [10], p. 35, Theorem 2.23), inequality (7) to estimate $c(s)$ from above and inequalities

$$\begin{aligned} \mathbb{E} \|\mathcal{G}(\cdot, t) - \mathcal{G}(\cdot, u)\|^4 &\leq K_{4,2}^4 (\mathbb{E} \|\mathcal{G}(\cdot, t) - \mathcal{G}(\cdot, u)\|^2)^2 \\ &< K_{4,2}^4 \left[\int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)]^2} \mu(dv) + \int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)][2d(v)-1]} \mu(dv) \right]^2 \cdot |t-u|^2 \end{aligned}$$

and

$$\mathbb{E} \|\mathcal{G}'(\cdot, t) - \mathcal{G}'(\cdot, u)\|^4 \leq K_{4,2}^4 \left[\int_{\mathbb{S}} \sigma^2(v) \mu(dv) \right]^2 |t-u|^2,$$

we conclude that the processes $\{\mathcal{G}(\cdot, t), t \in [0, 1]\}$ and $\{\mathcal{G}'(\cdot, t), t \in [0, 1]\}$ have continuous versions. \square

Passing to continuous versions, we thus consider Gaussian stochastic processes \mathcal{G} and \mathcal{G}' as Gaussian random elements in the space $C([0, 1]; L_2(\mu))$. Clearly \mathcal{G}' is an $L_2(\mu)$ -valued Wiener process.

Now we are ready to state our main results. As usual $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution.

Theorem 1. *Suppose that $d \in (1/2, 1)$, the integrals*

$$\mathbb{E} \left[\int_{\mathbb{S}} \frac{\varepsilon_0^2(v)}{[1-d(v)]^2} \mu(dv) \right]^{p/2} \quad \text{and} \quad \int_{\mathbb{S}} \frac{\sigma^2(v)}{[1-d(v)][2d(v)-1]} \mu(dv)$$

are finite and either $p = 2$ and $\bar{d} = \text{ess sup } d < 1$ or $p > 2$. Then we have that

$$n^{-H} \zeta_n \xrightarrow{\mathcal{D}} \mathcal{G} \quad \text{as } n \rightarrow \infty$$

in the space $C([0, 1]; L_2(\mu))$, where $\{n^{-H}\}$ is a sequence of multiplication operators given by $n^{-H}f = \{n^{-[3/2-d(s)]}f(s) : s \in \mathbb{S}\}$ for $f \in L_2(\mu)$.

Remark 6. We have that, for $d \in (1/2, 1)$ and $p > 0$,

$$\mathbb{E} \|\varepsilon_0\|^p < 2^{-p} \mathbb{E} \left[\int_{\mathbb{S}} \frac{\varepsilon_0^2(v)}{[1-d(v)]^2} \mu(dv) \right]^{p/2}$$

since $1-d(v) < 1/2$.

Theorem 2. *Suppose that $d = 1$ and $\mathbb{E} \|\varepsilon_0\|^p < \infty$ for some $p > 2$. Then we have that*

$$(\sqrt{n} \log n)^{-1} \zeta_n \xrightarrow{\mathcal{D}} \mathcal{G}' \quad \text{as } n \rightarrow \infty$$

in the space $C([0, 1]; L_2(\mu))$.

Theorem 3. *Suppose that $\underline{d} = \text{ess inf } d > 1$ and $\mathbb{E} \|\varepsilon_0\|^2 < \infty$. Then we have that*

$$(\sqrt{n})^{-1} \zeta_n \xrightarrow{\mathcal{D}} \mathcal{G}' \quad \text{as } n \rightarrow \infty$$

in the space $C([0, 1]; L_2(\mu))$.

Proof of Theorem 3. The convergence of Theorem 3 follows from Theorem 5 of Račkauskas and Suquet [18] since $\sum_{j=0}^{\infty} \|u_j\| = \sum_{j=1}^{\infty} j^{-\underline{d}} < \infty$. \square

4.2 Proof of Theorem 1 and Theorem 2

The proof contains two major parts. We prove the convergence of the finite-dimensional distributions of the sequences $\{n^{-H}\zeta_n\}$ and $\{(\sqrt{n}\log n)^{-1}\zeta_n\}$ in the first part and we prove the tightness of these sequences in the second part.

To avoid considerations of two separate but similar cases, we denote $b_n^{-1} = n^{-H}$ and $\zeta = \mathcal{G}$ in the proof of Theorem 1, whereas $b_n = \sqrt{n}\log n$ and $\zeta = \mathcal{G}'$ in the proof of Theorem 2.

Convergence of the finite-dimensional distributions

The convergence of the finite-dimensional distributions means that the convergence

$$\left(b_n^{-1}\zeta_n(t_1) \quad \dots \quad b_n^{-1}\zeta_n(t_q) \right) \xrightarrow{\mathcal{D}} \left(\zeta(t_1) \quad \dots \quad \zeta(t_q) \right) \quad (32)$$

holds in the space $L_2^q(\mu)$ for all $q \in \mathbb{N}$ and for all $t_1, \dots, t_q \in [0, 1]$. Note that the space $L_2^q(\mu)$ is isomorphic to $L_2(\mu; \mathbb{R}^q)$, the space of \mathbb{R}^q -valued square μ -integrable functions with the norm

$$\|f\| = \left[\int_{\mathbb{S}} \|f(v)\|^2 \mu(dv) \right]^{1/2}, \quad f \in L_2(\mu; \mathbb{R}^q),$$

where $\|f(v)\|$ denotes the Euclidean norm in \mathbb{R}^q .

Fix $t_1, \dots, t_q \in [0, 1]$ and denote, for $s \in \mathbb{S}$,

$$\zeta_n^{(q)}(s) = (\zeta_n(t_1, s), \dots, \zeta_n(t_q, s))^T \quad \text{and} \quad \zeta^{(q)}(s) = (\zeta(t_1, s), \dots, \zeta(t_q, s))^T,$$

where \mathbf{x}^T denotes transpose of a vector \mathbf{x} .

Let $\zeta_n^{(q)} = \{\zeta_n^{(q)}(s) : s \in \mathbb{S}\}$ and $\zeta^{(q)} = \{\zeta^{(q)}(s) : s \in \mathbb{S}\}$. We need to prove that

$$b_n^{-1}\zeta_n^{(q)} \xrightarrow{\mathcal{D}} \zeta^{(q)} \quad (33)$$

in the space $L_2(\mu; \mathbb{R}^q)$ to establish (32).

According to Theorem 2 in Cremers and Kadelka [4], it suffices to prove the following:

- (I) there exists a measurable set $\mathbb{S}_0 \subset \mathbb{S}$ such that $\mu(\mathbb{S} \setminus \mathbb{S}_0) = 0$ and for any $p \in \mathbb{N}$ and $s_1, \dots, s_p \in \mathbb{S}_0$ we have that

$$\left(b_n^{-1}\zeta_n^{(q)}(s_1) \quad \dots \quad b_n^{-1}\zeta_n^{(q)}(s_p) \right) \xrightarrow{\mathcal{D}} \left(\zeta^{(q)}(s_1) \quad \dots \quad \zeta^{(q)}(s_p) \right);$$

- (II) (a) for each $s \in \mathbb{S}$,

$$\mathbb{E} \|b_n^{-1}\zeta_n^{(q)}(s)\|^2 \rightarrow \mathbb{E} \|\zeta^{(q)}(s)\|^2;$$

- (b) there exists a μ -integrable function $f : \mathbb{S} \rightarrow [0, \infty)$ such that for each $s \in \mathbb{S}$ and each $n \in \mathbb{N}$

$$\mathbb{E} \|b_n^{-1}\zeta_n^{(q)}(s)\|^2 \leq f(s).$$

We use an auxiliary result to prove (I) which is stated in Lemma 1 below and may be explained as follows.

Let \mathbb{E} and \mathbb{F} be two separable Hilbert spaces and let $L(\mathbb{E}, \mathbb{F})$ be the space of bounded linear operators from \mathbb{E} to \mathbb{F} . Suppose that a sequence $\{Z_n\}$ of \mathbb{F} -valued random elements can be expressed as

$$Z_n = \sum_{j=-\infty}^{\infty} B_{nj} \xi_j,$$

where $\{B_{nj}\}$ is a sequence in $L(\mathbb{E}, \mathbb{F})$ for each $n \in \mathbb{N}$ and $\{\xi_j\}$ is a sequence of independent and identically distributed \mathbb{E} -valued random elements with $\mathbb{E} \xi_0 = 0$ and $\mathbb{E} \|\xi_0\|^2 < \infty$. Using the same linear bounded operators $\{B_{nj}\}$, we construct another sequence $\{\tilde{Z}_n\}$ of \mathbb{F} -valued random elements that can be represented as

$$\tilde{Z}_n = \sum_{j=-\infty}^{\infty} B_{nj} \tilde{\xi}_j,$$

where $\{\tilde{\xi}_j\}$ is a sequence of independent and identically distributed \mathbb{E} -valued Gaussian random elements with $\mathbb{E} \tilde{\xi}_0 = 0$ and the same covariance operator as that of ξ_0 .

Under the conditions of Lemma 1 below, the sequences $\{Z_n\}$ and $\{\tilde{Z}_n\}$ have the same limiting behaviour, i.e. if one converges in distribution then so does the other and their limits coincide. Before we state Lemma 1, we define the distance function ρ_k .

Definition. Let U and V be random elements with values in a separable Hilbert space \mathbb{H} . The distance function ρ_k is given by

$$\rho_k(U, V) = \sup_{f \in F_k} |\mathbb{E} f(U) - \mathbb{E} f(V)|,$$

where F_k is the set of k times Frechet differentiable functions $f : \mathbb{H} \rightarrow \mathbb{R}$ such that

$$\sup_{x \in \mathbb{H}} |f^{(j)}(x)| \leq 1, \quad j = 0, 1, \dots, k.$$

It is proved in the paper by Giné and León [7] that, for every $k > 0$, the distance function ρ_k metrizes the convergence in distribution of sequences of random elements with values in \mathbb{H} .

Lemma 1. *If both of the conditions*

$$\lim_{n \rightarrow \infty} \sup_{j \in \mathbb{Z}} \|B_{nj}\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup \sum_{j=-\infty}^{\infty} \|B_{nj}\|^2 < \infty \quad (34)$$

are satisfied, then $\lim_{n \rightarrow \infty} \rho_3(Z_n, \tilde{Z}_n) = 0$.

Proof. The proof follows from the proof of Proposition 4.1 of Račkauskas and Suquet [19]. The only difference is that $\mathbb{E} = \mathbb{F}$ in Račkauskas and Suquet [19], but the proof remains valid as long as

$$\|B_{nk}\| \leq \|B_{nk}\| \|f\|$$

for each $n \in \mathbb{N}$, each $k \in \mathbb{Z}$ and each $f \in \mathbb{E}$. □

Let $s_1, \dots, s_p \in \mathbb{S}$. We express the sequence $\{(b_n^{-1} \zeta_n^q(s_1) \quad \dots \quad b_n^{-1} \zeta_n^q(s_p))\}$ of random matrices as

$$\begin{aligned} (b_n^{-1} \zeta_n^q(s_1) \quad \dots \quad b_n^{-1} \zeta_n^q(s_p)) &= \sum_{j=-\infty}^{\infty} \begin{pmatrix} z_n^{-1}(s_1) a_{nj}(s_1, t_1) \varepsilon_j(s_1) & \cdots & z_n^{-1}(s_p) a_{nj}(s_p, t_1) \varepsilon_j(s_p) \\ \vdots & \ddots & \vdots \\ z_n^{-1}(s_1) a_{nj}(s_1, t_q) \varepsilon_j(s_1) & \cdots & z_n^{-1}(s_p) a_{nj}(s_p, t_q) \varepsilon_j(s_p) \end{pmatrix} \\ &= \sum_{j=-\infty}^{\infty} A_{nj} \mathcal{E}_j, \end{aligned}$$

where

$$A_{nj} = \begin{pmatrix} z_n^{-1}(s_1)a_{nj}(s_1, t_1) & \cdots & z_n^{-1}(s_p)a_{nj}(s_p, t_1) \\ \vdots & \ddots & \vdots \\ z_n^{-1}(s_1)a_{nj}(s_1, t_q) & \cdots & z_n^{-1}(s_p)a_{nj}(s_p, t_q) \end{pmatrix}, \quad \mathcal{E}_j = \text{diag}(\varepsilon_j(s_1) \quad \dots \quad \varepsilon_j(s_p))$$

and

$$z_n(s) = \begin{cases} n^{3/2-d(s)}, & \text{if } 1/2 < d(s) < 1; \\ \sqrt{n} \log n, & \text{if } d(s) = 1. \end{cases}$$

If $d \in (1/2, 1]$, then the matrices $\{A_{nj}\}$ satisfy both of conditions (34). Indeed, since

$$\sup_{j \in \mathbb{Z}} a_{nj}(s, t) = a_{n1}(s, t) = \sum_{k=1}^{\lfloor nt \rfloor} k^{-d(s)} + \{nt\}(\lfloor nt \rfloor + 1)^{-d(s)},$$

we have the following asymptotic relations

$$\sup_{j \in \mathbb{Z}} a_{nj}(s, t) \sim \begin{cases} \frac{t^{1-d(s)}}{1-d(s)} \cdot n^{1-d(s)}, & \text{if } d(s) < 1; \\ \log n, & \text{if } d(s) = 1. \end{cases}$$

We have that

$$\mathbb{E} \zeta_n^2(s, t) = \sigma^2(s) \sum_{j=-\infty}^{\lfloor nt \rfloor + 1} a_{nj}^2(s, t)$$

and we use the asymptotic behaviour of the variance $\mathbb{E} \zeta_n^2(s, t)$ (see Remark 5) to obtain the following asymptotic relations

$$\sum_{j=-\infty}^{\lfloor nt \rfloor + 1} a_{nj}^2(s, t) \sim \begin{cases} \frac{c(s)}{[1-d(s)][3-2d(s)]} \cdot t^{3-2d(s)} \cdot n^{3-2d(s)}, & \text{if } 1/2 < d(s) < 1; \\ t \cdot n \log^2 n, & \text{if } d(s) = 1. \end{cases}$$

Now we investigate the sequence $\{(b_n^{-1} \tilde{\zeta}_n^q(s_1) \quad \dots \quad b_n^{-1} \tilde{\zeta}_n^q(s_p))\}$, which is expressed as

$$(b_n^{-1} \tilde{\zeta}_n^q(s_1) \quad \dots \quad b_n^{-1} \tilde{\zeta}_n^q(s_p)) = \sum_{j=-\infty}^{\infty} A_{nj} \tilde{\mathcal{E}}_j, \quad (35)$$

where $\{\tilde{\mathcal{E}}_j\}$ is a sequence of independent and identically distributed Gaussian random matrices with zero mean and the same covariance operator as that of \mathcal{E}_0 . Since $\{(\tilde{\zeta}_n^q(r) \quad \dots \quad \tilde{\zeta}_n^q(s_p))\}$ is a sequence of finite-dimensional Gaussian random elements, we only need to check for each $i = 1, \dots, p$ and each $j = 1, \dots, q$ the convergence

$$z_n^{-1}(s_i) \mathbb{E} \zeta_n(s_i, t_j) \rightarrow \mathbb{E} \zeta(s_i, t_j).$$

But this easily follows from Proposition 5 and Proposition 6. The proof of (I) is complete.

Next we prove (II). We prove (IIa) using equalities

$$\mathbb{E} \|b_n^{-1} \zeta_n^q(s)\|^2 = \sum_{i=1}^q \mathbb{E} [z_n^{-1}(s) \zeta_n(s, t_i)]^2 \quad \text{and} \quad \mathbb{E} \|\zeta^q(s)\|^2 = \sum_{i=1}^q \mathbb{E} \zeta^2(s, t_i)$$

and Remark 5.

An auxiliary result is needed to prove part (IIb).

Proposition 11. *If $1/2 < d(s) < 1$, then*

$$\mathbb{E}[n^{-[3/2-d(s)]}\zeta_n(s, t)]^2 \leq g(s) = 2[g_1(s) + g_2(s) + g_3(s)], \quad (36)$$

for each $n \in \mathbb{N}$, where

$$g_1(s) = \sigma^2(s) \left[1 + \frac{1}{2d(s) - 1} \right], \quad g_2(s) = \frac{\sigma^2(s)}{[1 - d(s)]^2}, \quad g_3(s) = \frac{\sigma^2(s)}{[1 - d(s)][2d(s) - 1]}$$

and $c(s)$ is given by (6).

If $d = 1$, then

$$\mathbb{E}[(\sqrt{n} \log n)^{-1} \zeta_n(s, t_i)]^2 \leq M \cdot \sigma^2(s), \quad (37)$$

where M is a positive constant.

Proof. Expanding $\mathbb{E} \zeta_n^2(s, t)$ gives

$$\mathbb{E} \zeta_n^2(s, t) = [nt] \gamma_0(s) + 2 \sum_{k=1}^{[nt]} ([nt] - k) \gamma_k(s) + 2 \{nt\} \sum_{k=1}^{[nt]} \gamma_k(s) + \{nt\}^2 \gamma_0(s). \quad (38)$$

Using expression (4) for cross-covariance, bounding series with integrals from above and using inequality (7) leads to the following inequalities that complete the proof of inequality (36):

$$\gamma_0(s) \leq \sigma^2(s) \left[1 + \frac{1}{2d(s) - 1} \right],$$

$$\sum_{k=1}^{[nt]} ([nt] - k) \gamma_k(s) \leq \frac{1}{2} \left[\frac{\sigma^2(s)}{[1 - d(s)]^2} + \frac{\sigma^2(s)}{[1 - d(s)][2d(s) - 1]} \right] [nt]^{3-2d(s)}$$

and

$$\sum_{k=1}^{[nt]} \gamma_k(s) \leq \frac{1}{2} \left[\frac{\sigma^2(s)}{[1 - d(s)]^2} + \frac{\sigma^2(s)}{[1 - d(s)][2d(s) - 1]} \right] [nt]^{2[1-d(s)]}.$$

We argue as follows to prove inequality (37). By setting $r = s$ in expression (4), we see that the only term in expression (38) that depends on s is $\sigma^2(s)$ since $d(s) = 1$ for each $s \in \mathbb{S}$. It follows that the sequence

$$\frac{1}{\sigma^2(s)} \cdot \mathbb{E}[(\sqrt{n} \log n)^{-1} \zeta_n(s, t)]^2$$

does not depend on s and it is a convergent sequence (see Remark 5). So it is bounded by some positive constant, say M . \square

Now we can obtain the required function f using Proposition 11, the fact that $\mathbb{E} \|\zeta^q(s)\|^2 = \sum_{i=1}^q \mathbb{E} \zeta^2(s, t_i)$ and setting

$$f(s) = \begin{cases} q \cdot g(s), & \text{if } d(s) < 1; \\ qM \cdot \sigma^2(s), & \text{if } d(s) = 1. \end{cases}$$

The proof of (II) is complete. This completes the proof of the convergence of the finite dimensional distributions of the sequence $\{b_n^{-1} \zeta_n\}$.

Tightness

To establish tightness of the sequence $\{b_n^{-1}\zeta_n\}$, we use the following adaptation of Theorem 12.3 from Billingsley [1] (see also Proposition 4.2 in Račkauskas and Suquet [19]).

Proposition 12. *Let \mathbb{H} be a separable Hilbert space. The sequence $\{Z_n\}$ of random elements of the space $C([0, 1]; \mathbb{H})$ is tight if*

- (i) $\{Z_n(t)\}$ is tight on \mathbb{H} for every $t \in [0, 1]$;
- (ii) there exists $\gamma \geq 0$, $a > 1$ and a continuous increasing function $F : [0, 1] \rightarrow \mathbb{R}$ such that

$$P(\|Z_n(t) - Z_n(u)\| > \lambda) \leq \lambda^{-\gamma} |F(t) - F(u)|^a.$$

It follows from Characiejus and Račkauskas [2] that the sequence $\{b_n^{-1}S_n\}$ converges in distribution in $L_2(\mu)$. Hence the sequence $\{b_n^{-1}\zeta_n(t)\}$ also converges in distribution in $L_2(\mu)$ and the sequence $\{b_n^{-1}\zeta_n(t)\}$ is tight on $L_2(\mu)$ for every $t \in [0, 1]$ and condition (i) of Proposition 12 holds.

Now we show that condition (ii) of Proposition 12 holds for the sequence $\{b_n^{-1}\zeta_n\}$. By C we denote a generic positive constant, not necessarily the same at different occurrences. We also denote

$$\Delta_n^p(t, u) = \mathbb{E} \|b_n^{-1}[\zeta_n(t) - \zeta_n(u)]\|^p,$$

where $p \geq 2$, $t, u \in [0, 1]$ and $n \geq 1$.

Proposition 13. *Suppose that $d \in (1/2, 1)$ and the integrals*

$$\int_{\mathbb{S}} \frac{\sigma^2(r)}{2d(r) - 1} \mu(dr) \quad \text{and} \quad \mathbb{E} \left[\int_{\mathbb{S}} \frac{\varepsilon_0^2(r)}{[1 - d(r)]^2} \mu(dr) \right]^{p/2}, \quad p \geq 2,$$

are finite. Let $\bar{d} = \text{ess sup } d$. Then

$$\Delta_n^p(t, u) \leq C \cdot |t - u|^{(3-2\bar{d})p/2}, \quad n \geq 1. \quad (39)$$

Suppose that $d = 1$ and $\mathbb{E} \|\varepsilon_0\|^p < \infty$ for $p \geq 2$. Then

$$\Delta_n^p(t, u) \leq C \cdot |t - u|^{p/2}, \quad n \geq 2. \quad (40)$$

We recall that $\lfloor \cdot \rfloor$ is the floor function defined by $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$ for $x \in \mathbb{R}$, $\lceil \cdot \rceil$ is the ceiling function defined by $\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}$ for $x \in \mathbb{R}$ and $\{x\} = x - \lfloor x \rfloor$ is a fractional part of $x \in \mathbb{R}$. Observe that $\{x\} = 0$ if and only if $x \in \mathbb{Z}$ and

$$\lceil x \rceil - \lfloor x \rfloor = \begin{cases} 0, & \text{if } x \in \mathbb{Z}; \\ 1, & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}. \end{cases}$$

We need an auxiliary lemma to prove Proposition 13.

Lemma 2. *Let $0 \leq u < t \leq 1$, $n \geq 1$ and $\{nt\} = \{nu\} = 0$.*

If $d \in (1/2, 1)$, then

$$n^{-[3-2d(s)]} \sum_{j=-\infty}^{nt} \left[\sum_{k=nu+1}^{nt} v_{k-j}(s) \right]^2 \leq \left[\frac{2}{[1-d(s)]^2} + \frac{1}{2d(s)-1} \right] \cdot |t - u|^{3-2d(s)} \quad (41)$$

for $n \geq 1$, where $v_j(s)$ is given by (11).

If $d = 1$, then

$$(\sqrt{n} \log n)^{-p} \sum_{j=-\infty}^{nt} \left[\sum_{k=nu+1}^{nt} v_{k-j} \right]^p \leq C \cdot |t - u|^{p/2}, \quad (42)$$

for $n \geq 2$ and $p \geq 2$, where v_j is given by (9).

Proof. We investigate the series

$$\sum_{j=-\infty}^{nt} \left[\sum_{k=nu+1}^{nt} v_{k-j}(s) \right]^p \quad (43)$$

with $p = 2$ in the case of $d \in (1/2, 1)$ and $p \geq 2$ in the case of $d = 1$. Let us split series (43) into two terms

$$\sum_{j=-\infty}^{nt} \left[\sum_{k=nu+1}^{nt} v_{k-j}(s) \right]^p = \sum_{j=-nu+1}^{\infty} \left[\sum_{k=nu+1}^{nt} (k+j)^{-d(s)} \right]^p + \sum_{j=nu+1}^{nt} \left[\sum_{k=1}^{nt-j+1} k^{-d(s)} \right]^p \quad (44)$$

and then split the first term on the right-hand side of (44) again into two terms

$$\begin{aligned} \sum_{j=-nu+1}^{\infty} \left[\sum_{k=nu+1}^{nt} (k+j)^{-d(s)} \right]^p &= \sum_{j=-nu+1}^{n(t-2u)} \left[\sum_{k=nu+1}^{nt} (k+j)^{-d(s)} \right]^p \\ &\quad + \sum_{j=n(t-2u)+1}^{\infty} \left[\sum_{k=nu+1}^{nt} (k+j)^{-d(s)} \right]^p. \end{aligned} \quad (45)$$

The first term on the right-hand side of (45) is estimated from above in the following way:

$$n^{-[3-2d(s)]} \sum_{j=-nu+1}^{n(t-2u)} \left[\sum_{k=nu+1}^{nt} (k+j)^{-d(s)} \right]^2 \leq \frac{n|t-u|}{n^{3-2d(s)}} \left[\sum_{k=nu+1}^{nt} (k-nu)^{-d(s)} \right]^2 \leq \frac{|t-u|^{3-2d(s)}}{[1-d(s)]^2}$$

if $d \in (1/2, 1)$;

$$\begin{aligned} (\sqrt{n} \log n)^{-p} \sum_{j=-nu+1}^{n(t-2u)} \left[\sum_{k=nu+1}^{nt} (k+j)^{-1} \right]^p &\leq \left[\frac{1}{\log n} \sum_{k=nu+1}^{nt} (k-nu)^{-1} \right]^p \cdot |t-u|^{p/2} \\ &\leq \left[\frac{1 + \log(n|t-u|)}{\log n} \right]^p \cdot |t-u|^{p/2} \end{aligned}$$

if $d = 1$ since $1/n \leq |t-u|$ (otherwise $t = u$ because we assume that $\{nt\} = \{nu\} = 0$). The second term on the right-hand side of (45) is estimated from above using the inequality

$$\sum_{j=n(t-2u)+1}^{\infty} \left[\sum_{k=nu+1}^{nt} (k+j)^{-d(s)} \right]^p \leq (n|t-u|)^p \sum_{j=n(t-2u)+1}^{\infty} (nu+j)^{-pd(s)} \leq \frac{(n|t-u|)^{p+1-pd(s)}}{pd(s)-1}$$

and observing that

$$n^{-[3-2d(s)]} (n|t-u|)^{p+1-pd(s)} = |t-u|^{3-2d(s)}$$

if $d \in (1/2, 1)$ and $p = 2$ and

$$(\sqrt{n} \log n)^{-p} (n|t-u|)^{p+1-pd(s)} \leq \frac{|t-u|^{p/2}}{\log^p 2}$$

if $d = 1$ and $p \geq 2$ since $1/n \leq |t-u|$.

The second term on the right-hand side of (44) is estimated in the following way:

$$n^{-[3-2d(s)]} \sum_{j=nu+1}^{nt} \left[\sum_{k=1}^{nt-j+1} k^{-d(s)} \right]^2 \leq \frac{1}{[1-d(s)]^2 [3-2d(s)]} \cdot |t-u|^{3-2d(s)}$$

if $d \in (1/2, 1)$ and

$$\begin{aligned} (\sqrt{n} \log n)^{-p} \sum_{j=nu+1}^{nt} \left[\sum_{k=1}^{nt-j+1} k^{-1} \right]^p &\leq \frac{1}{n^{p/2} \log^p n} \sum_{j=nu+1}^{nt} [1 + \log(nt - j + 1)]^p \\ &\leq 2 \left[\frac{1}{\log 2} + \frac{\log(n|t - u|)}{\log n} \right]^p \cdot |t - u|^{p/2} \end{aligned}$$

if $d = 1$. The proof of Lemma 2 is complete. \square

Now we are ready to prove Proposition 13.

Proof of Proposition 13. Let $t, u \in [0, 1]$. There is no loss of generality by assuming that $t > u$. Set $t' = \lfloor nt \rfloor / n$ and $u' = \lceil nu \rceil / n$, so that $t, t' \in [\lfloor nt \rfloor / n, \lceil nt \rceil / n]$, $u, u' \in [\lfloor nu \rfloor / n, \lceil nu \rceil / n]$, $\{nt'\} = \{nu'\} = 0$ and $|t' - u'| \leq |t - u|$. Since

$$\Delta_n^p(t, u) \leq C[\Delta_n^p(t, t') + \Delta_n^p(t', u') + \Delta_n^p(u', u)],$$

we can establish inequalities (39) and (40) by investigating two cases: either $t, u \in [\kappa/n, (\kappa + 1)/n]$ for some $\kappa \in \{0, \dots, n - 1\}$ or $\{nt\} = \{nu\} = 0$.

First, suppose that $t, u \in [\kappa/n, (\kappa + 1)/n]$ for some $\kappa \in \{0, \dots, n - 1\}$. Then $|t - u| \leq 1/n$ and

$$\zeta_n(t) - \zeta_n(u) = n|t - u|X_{\kappa+1},$$

so that

$$\Delta_n^p(t, u) \leq [n|t - u|]^p \|n^{-H}\|^p \mathbb{E} \|X_0\|^p \leq \mathbb{E} \|X_0\|^p \cdot |t - u|^{(3-2d)p/2}$$

if $d \in (1/2, 1)$ and

$$\Delta_n^p(t, u) = [n|t - u|]^p (\sqrt{n} \log n)^{-p} \mathbb{E} \|X_0\|^p \leq \frac{\mathbb{E} \|X_0\|^p}{\log^p 2} \cdot |t - u|^{p/2}$$

if $d = 1$ and $n \geq 2$.

We have that

$$\mathbb{E} \|X_0\|^p \leq 2^{p-1} \left(C \frac{p}{\log p} \right)^p \left[(\mathbb{E} \|X_0\|^2)^{p/2} + \sum_{j=0}^{\infty} \mathbb{E} \|u_j \varepsilon_{k-j}\|^p \right] \quad (46)$$

by using a slight modification of the inequality stated in Theorem 6.20 of Ledoux and Talagrand [14]. Since

$$\mathbb{E} \|X_0\|^2 \leq \mathbb{E} \|\varepsilon_0\|^2 + \int_{\mathbb{S}} \frac{\sigma^2(r)}{2d(r) - 1} \mu(dr)$$

and $\sum_{j=0}^{\infty} \mathbb{E} \|u_j \varepsilon_{k-j}\|^p \leq \mathbb{E} \|\varepsilon_0\|^p \sum_{j=1}^{\infty} j^{-p/2}$, we have that $\mathbb{E} \|X_0\|^p < \infty$.

Secondly, suppose that $\{nt\} = \{nu\} = 0$. Then $|t - u| \geq 1/n$ ($\Delta_n^p(t, u) = 0$ if $\{nt\} = \{nu\} = 0$ and $|t - u| < 1/n$). The increment $b_n^{-1}[\zeta_n(t) - \zeta_n(u)]$ may be expressed as a series of independent $L_2(\mu)$ -valued random elements

$$b_n^{-1}[\zeta_n(t) - \zeta_n(u)] = \sum_{j=-\infty}^{nt} b_n^{-1} \sum_{k=nu+1}^{nt} v_{k-j} \varepsilon_j$$

where v_j is given by (11). Using the same inequality as in (46), we have that

$$\Delta_n^p(t, u) \leq 2^{p-1} \left(C \frac{p}{\log p} \right)^p \left[\Delta_n^{p/2}(t, u) + \sum_{j=-\infty}^{nt} \mathbb{E} \left\| b_n^{-1} \sum_{k=nu+1}^{nt} v_{k-j} \varepsilon_j \right\|^p \right].$$

If $d \in (1/2, 1)$, then we have that

$$\Delta_n^2(t, u) = \int_{\mathbb{S}} \sigma^2(r) n^{-[3-2d(r)]} \sum_{j=-\infty}^{nt} \left[\sum_{k=nu+1}^{nt} v_{k-j}(r) \right]^2 \mu(dr) \quad (47)$$

and

$$\sum_{j=-\infty}^{nt} \mathbb{E} \left\| b_n^{-1} \sum_{k=nu+1}^{nt} v_{k-j} \varepsilon_j \right\|^p = \sum_{j=-\infty}^{nt} \mathbb{E} \left[\int_{\mathbb{S}} n^{-[3-2d(r)]} \left| \sum_{k=nu+1}^{nt} v_{k-j}(r) \right|^2 \varepsilon_j^2(r) \mu(dr) \right]^{p/2}. \quad (48)$$

If $d = 1$, then we obtain

$$\Delta_n^2(t, u) = \mathbb{E} \|\varepsilon_0\|^2 (\sqrt{n} \log n)^{-2} \sum_{j=-\infty}^{nt} \left[\sum_{k=nu+1}^{nt} v_{k-j} \right]^2 \quad (49)$$

and

$$\sum_{j=-\infty}^{nt} \mathbb{E} \left\| b_n^{-1} \sum_{k=nu+1}^{nt} v_{k-j} \varepsilon_j \right\|^p t = \mathbb{E} \|\varepsilon_0\|^p (\sqrt{n} \log n)^{-p} \sum_{j=-\infty}^{nt} \left[\sum_{k=nu+1}^{nt} v_{k-j} \right]^p. \quad (50)$$

We estimate (47), (49) and (50) using Lemma 2 and we need to estimate series (48) for $p > 2$ when $d \in (1/2, 1)$. As in (44) and (45), we split series (48) into three parts and estimate them from above separately. The estimation is essentially similar to the estimation of series (43). Let us recall that we assume that $1/n \leq |t - u|$ if $\{nt\} = \{nu\} = 0$. The following inequalities are obtained:

$$\sum_{j=-nu+1}^{n(t-2u)} \mathbb{E} \left[\int_{\mathbb{S}} \frac{|\sum_{k=nu+1}^{nt} (k+j)^{-d(r)}|^2 \varepsilon_j^2(r)}{n^{3-2d(r)}} \mu(dr) \right]^{p/2} \leq \mathbb{E} \left[\int_{\mathbb{S}} \frac{\varepsilon_0^2(r)}{[1-d(r)]^2} \mu(dr) \right]^{p/2} |t-u|^{(3-2\bar{d})p/2}$$

since

$$\frac{\sum_{k=nu+1}^{nt} (k-nu)^{-d(r)}}{n^{1-d(r)}} \leq \frac{|t-u|^{1-d(r)}}{1-d(r)};$$

$$\sum_{j=n(t-2u)+1}^{\infty} \mathbb{E} \left[\int_{\mathbb{S}} \frac{|\sum_{k=nu+1}^{nt} (k+j)^{-d(r)}|^2 \varepsilon_j^2(r)}{n^{3-2d(r)}} \mu(dr) \right]^{p/2} \leq \frac{\mathbb{E} \|\varepsilon_0\|^p}{p/2-1} |t-u|^{(3-2\bar{d})p/2}$$

since

$$\left(\frac{n}{nu+j} \right)^{2d(r)} = \left(\frac{n}{n|t-u|} \right)^{2d(r)} \left(\frac{n|t-u|}{nu+j} \right)^{2d(r)} \leq n|t-u|^{1-2\bar{d}} (nu+j)^{-1}$$

for $j \geq n(t-2u)+1$;

$$\sum_{j=nu+1}^{nt} \mathbb{E} \left[\int_{\mathbb{S}} \frac{|\sum_{k=1}^{nt-j+1} k^{-d(r)}|^2 \varepsilon_j^2(r)}{n^{3-2d(r)}} \mu(dr) \right]^{p/2} \leq \frac{2^{1+p(1-\bar{d})}}{1+p(1-\bar{d})} \mathbb{E} \left[\int_{\mathbb{S}} \frac{\varepsilon_0^2(r)}{[1-d(r)]^2} \mu(dr) \right]^{p/2} |t-u|^{(3-\bar{d})p/2}$$

since

$$\frac{\sum_{k=1}^{nt-j+1} k^{-d(r)}}{n^{1-d(r)}} \leq \frac{1}{1-d(r)} \left[\frac{nt-j+1}{n} \right]^{1-d(r)} \leq \frac{1}{1-d(r)} \left[\frac{nt-j+1}{n} \right]^{1-\bar{d}}$$

for $nu+1 \leq j \leq nt$.

The proof of Proposition 13 is complete. \square

We established the convergence of the finite-dimensional distributions and the tightness of the sequence $\{b_n^{-1}\zeta_n\}$. The proof of Theorem 1 and Theorem 2 is complete.

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